International Journal of Theoretical Physics, Vol. 29, No. 3, 1990

Möbius Groups over General Fields Using Clifford Algebras Associated with Spheres

Jay P. Fillmore[1] **and A. Springer**[2]

Received July 5, 1989

The space of 2-by-2 Hermitian matrices is isometric to Minkowski space. This is commonly used to exhibit the group $SL(2, \mathbb{C})$ as a twofold cover of the identity component of the Lorentz group. That these Hermitian matrices also represent equations of circles in the Euclidean plane leads to the group $PSL(2, \mathbb{C})$ as the Möbius group of the Euclidean plane. Clifford algebras naturally arise in the construction of covers of the orthogonal group by spin groups. By considering in addition the Clifford algebra of the space of equations of spheres, we are able to extend these ideas to the Möbius group of finite-dimensional vector spaces over general fields.

OVERVIEW

Representations of the Lorentz group $O(1, 3)$, with signature $(+---)$, have been of great interest to both physicists and mathematicians since Einstein introduced relativity. With the advent of quantum mechanics, and the Dirac equation in particular (Dirac, 1927, 1928), the spinor representation of the identity component $SO^+(1, 3)$ by use of its twofold cover $SL(2, \mathbb{C})$ has gained special importance. It arises as the "adjoint" representation of $SL(2, \mathbb{C})$ on the space of Hermitian matrices $H(2, \mathbb{C})$ (Penrose and Rindler, 1984). This space, equipped with the determinant as quadratic form, is isometric to Minkowski space, $\mathbb{R}^{1,3}$. What is less familiar is that $H(2, \mathbb{C})$ represents equations of "circles" in the Euclidean plane, and that the quadric in the projective space based on $H(2, \mathbb{C})$ is a conformal compactification of this plane (Hua, 1981). The Möbius group, which is a group of complex point transformations that sends "circles" to "circles," naturally acts on this quadric. The matrices of determinant zero, which represent points of this quadric, factor into a column $\binom{\xi}{\eta}$ in $\mathbb{C}^2$ and a row which is the Hermitian

[1]Department of Mathematics, University of California at San Diego, LaJolla, California.
[2]Department of Natural Science, San Diego State University, San Diego, California.

0020-7748/90/0300-0225$06.00/0

adjoint $\binom{\xi}{\eta}^{t*}$. It is these columns, called spinors, on which $SL(2,\mathbb{C})$ naturally acts. The group $SL(2,\mathbb{C})$ also has a projective spinor representation when it acts on the complex projective line consisting of points $\binom{\xi}{\eta}\mathring{\mathbb{C}}$, where $\mathring{\mathbb{C}}$ denotes the multiplicative group of $\mathbb{C}$. In analogy with classical projective geometry, we call $\binom{\xi}{\eta}$ the homogeneous coordinates of the projective spinor $\binom{\xi}{\eta}\mathring{\mathbb{C}}$.

In terms of Clifford algebras and their even subalgebras, the Clifford algebra based on $\mathbb{R}^{1,3}$ has its even subalgebra isomorphic to the algebra of all 2-by-2 complex matrices. The subgroup of the even Clifford group consisting of elements of unit spinorial norm is isomorphic to $SL(2,\mathbb{C})$. The Clifford algebra based on $\mathbb{R}^{0,2}$ has its even subalgebra isomorphic to $\mathbb{C}$. The even Clifford group is isomorphic to $\mathring{\mathbb{C}}$. Thus, we have an even Clifford algebra that consists of 2-by-2 matrices over another even Clifford algebra. The former is related to the space of equations of "circles," and the latter to the space of points. We will call this the classical case. See Lam (1973) for a general discussion of Clifford algebras.

In more detail, the equation of a "circle" is given by

$$pz^*z-\beta^*z-z^*\beta+q=0,$$

where p and q are real numbers and β is complex. To represent a proper circle, p cannot be zero and $\beta^*\beta$ must be greater than pq. This equation can be represented by a Hermitian matrix

$$\begin{pmatrix} p & -\beta \\ -\beta^* & q \end{pmatrix}$$

If it has determinant zero, it can be considered to represent a point in the Euclidean plane. Such matrices are of rank one and so have a factorization

$$\begin{pmatrix} p & -\beta \\ -\beta^* & q \end{pmatrix}=\begin{pmatrix}\xi\\ \eta\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}^{t*}\lambda$$

where ξ and η are complex numbers, λ is real, and $t*$ denotes the Hermitian adjoint. If η is not zero, then

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix}\mathring{\mathbb{C}}=\begin{pmatrix}z\\ 1\end{pmatrix}\mathring{\mathbb{C}}$$

and the matrix represents the point associated with the complex number z. Now the action

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix}\mathring{\mathbb{C}}\rightsquigarrow\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\mathring{\mathbb{C}}$$

induces the action

$$\begin{pmatrix} p & -\beta \\ -\beta^* & q \end{pmatrix} \mathring{\mathbb{C}} \rightsquigarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & -\beta \\ -\beta^* & q \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{t*} \mathring{\mathbb{C}}$$

which we call the projective vector action.

We here extend the classical case to the case of a finite-dimensional vector space over an arbitrary field of characteristic not two and equipped with a nondegenerate quadratic form, with these differences:

1. We will find the cover of the full orthogonal group by using the whole Clifford algebra.
2. The equation of a "sphere" will be determined by a quadratic form instead of by the Hermitian form associated with complex numbers.

Our approach is quite different from that of Vahlen as revived recently by Ahlfors and others (Ahlfors, 1986; Lounesto and Latvamaa, 1980; Lounesto and Springer, 1989).

1. SPACES, ALGEBRAS, SPHERES, AND SPINORS

Some of the terminology already used in the previous section will now be formally defined. The main aim will be to introduce notation used later.

Let $\mathbf{x}$ denote a vector in a finite-dimensional vector space X over an arbitrary commutative field K of characteristic not two. Let $\mathring{K}$ denote the multiplicative group of K. Equip X with a nondegenerate quadratic form, denote its associated bilinear form by $\mathbf{x}\cdot\mathbf{y}$, and denote its orthogonal group by $O(X)$. The Clifford algebra A associated with this quadratic form is the associative algebra generated by the vectors of X with relations $\mathbf{x}^2 = \mathbf{x}\cdot\mathbf{x}$ which imply $\mathbf{xy}+\mathbf{yx} = 2\mathbf{x}\cdot\mathbf{y}$. We will call the submonoid generated by the vectors of X the Clifford monoid and denote it by M. The subgroup generated by the nonisotropic vectors of X is called the Clifford group and is denoted by Γ.

The Clifford group Γ is a central extension of the orthogonal group $O(X)$, the nonisotropic vector $\mathbf{a}$ mapping to the reflection

$$\mathbf{x} \rightsquigarrow -\mathbf{axa}/\mathbf{a}\cdot\mathbf{a} = \mathbf{x} - \mathbf{a}(2\mathbf{a}\cdot\mathbf{x}/\mathbf{a}\cdot\mathbf{a})$$

in the space orthogonal to $\mathbf{a}$. See Conway *et al.* (1985).

Orthogonal transformations of X extend uniquely to automorphisms and antiautomorphisms of the Clifford algebra. In particular, the identity on X extends to the identity on A as well as to the anti-involution $\alpha \rightsquigarrow {}^J\alpha$; while negation on X, that is, $\mathbf{x} \rightsquigarrow -\mathbf{x}$, extends to the involution $\alpha \rightsquigarrow {}^I\alpha$ as well as to the anti-involution $\alpha \rightsquigarrow {}^*\alpha$ (Lam, 1973).

The equation of a "sphere" of X has the formula

$$w^0\mathbf{x}\cdot\mathbf{x} - 2\mathbf{w}\cdot\mathbf{x} + w^\infty = 0$$

where w^0 and w^∞ are scalars in K and $\mathbf{w}$ is a vector in X. The notation extends the usual polyspherical coordinate notation (Klein, 1926). This includes proper spheres ($w^0 \neq 0$ and $-w^0w^\infty + \mathbf{w}\cdot\mathbf{w} \neq 0$), proper hyperplanes ($w^0 = 0$ and $-w^0w^\infty + \mathbf{w}\cdot\mathbf{w} \neq 0$), and "cones" ($-w^0w^\infty + \mathbf{w}\cdot\mathbf{w} = 0$). Only the choice w^0, w^∞, and $\mathbf{w}$ equal to zero does not represent the equation of a "sphere." We represent the equation of a "sphere" by the matrix

$$\underset{\sim}{w} = \begin{pmatrix} \mathbf{w} & -w^\infty \\ w^0 & -\mathbf{w} \end{pmatrix}$$

Such matrices, plus the zero matrix, form a vector space $\underset{\sim}{W}$ over K of dimension two greater than that of X. All matrices obtained by multiplying by a nonzero scalar represent the same "sphere." The space W of "spheres" is then the projective space of one-dimensional subspaces of $\underset{\sim}{W}$.

Inspired by the role that $-w^0w^\infty + \mathbf{w}\cdot\mathbf{w}$ plays in distinguishing different types of "spheres," extend the quadratic form on X to $\underset{\sim}{W}$ by

$$\underset{\sim}{w} \rightsquigarrow \underset{\sim}{w}\cdot\underset{\sim}{w} = -w^0w^\infty + \mathbf{w}\cdot\mathbf{w}$$

If $w^0 = 1$ and $\underset{\sim}{w}\cdot\underset{\sim}{w} \neq 0$, the sphere is proper and $\underset{\sim}{w}\cdot\underset{\sim}{w}$ is the "square of its radius." The quadric of W, denoted by Ψ, is the subset consisting of "cones."

The Clifford algebra $\underset{\sim}{A}$ associated to the quadratic form of $\underset{\sim}{W}$ consists of two-by-two matrices over A, and is in fact all such matrices. Let $\underset{\sim}{\Gamma}$ denote its Clifford group. Also let $\bar{A}$ denote the set of all two-element columns over A, so $\underset{\sim}{A}$ acts on $\bar{A}$ by matrix multiplication.

The involutions and anti-involutions of $\underset{\sim}{A}$ will be denoted by the same symbols as those of A. For instance, ${}^J\underset{\sim}{a}$ in $\underset{\sim}{A}$ corresponds to ${}^Ja$ in A. Let $\underset{\sim}{\theta}$ denote the matrix

$$\begin{pmatrix} \mathbf{0} & 1 \\ 1 & \mathbf{0} \end{pmatrix}$$

Then ${}^J\underset{\sim}{a} = \underset{\sim}{\theta}\,{}^{t*}\underset{\sim}{a}\,\underset{\sim}{\theta}^{-1}$, where $t*$ transposes the matrix $\underset{\sim}{a}$ and applies $*$ to each entry. Note, in comparison to the classical case, that we find it convenient to write adjoints and transposes to the upper left. The anti-involution J induces a "Hermitian adjoint" j that sends a column $\bar{a}$ to the row ${}^j\bar{a} = {}^{t*}\bar{a}\underset{\sim}{\theta}^{-1}$. The Hermitian adjoint j and the anti-involutions J on $\underset{\sim}{A}$ and $*$ on A are related by

$${}^J(\bar{b}\,{}^j\bar{a}) = \bar{a}\,{}^j\bar{b}, \qquad {}^*({}^j\bar{b}\bar{a}) = {}^j\bar{a}\bar{b}, \qquad {}^j(\underset{\sim}{\alpha}\bar{b}\gamma) = {}^*\gamma\,{}^j\bar{b}\,{}^J\underset{\sim}{\alpha}$$

The latter replaces sesquilinear with "bi-Hermitian" with respect to the left $\underset{\sim}{A}$- right A-bimodule structure on $\bar{A}$.

Define $\bar{S}$ to be the subset of $\bar{A}$ consisting of columns

$$\bar{s} = \begin{pmatrix} s^1 \\ s^0 \end{pmatrix}$$

satisfying

1. s^1 and s^0 are the products of vectors in X.
2. $s^1 * s^0$ is a vector in X.
3. If neither s^1 nor s^0 is invertible, then $s^1 * s^0$ must be a nonzero isotropic vector of X.

Let S be the set of equivalent classes of $\bar{S}$ modulo the Clifford group Γ. We denote a typical element of S by $s = \bar{s}\Gamma$ and call $\bar{s}$ the homogeneous coordinates of the projective spinor s. The Clifford group $\underset{\sim}{\Gamma}$ acts on the space $\underset{\sim}{W}$ by the vector action $\underset{\sim}{w} \rightsquigarrow \gamma \circ \underset{\sim}{w} = \gamma \underset{\sim}{w}\,{}^J\gamma$ and on the projective quadric Ψ by the projective vector action $\psi = \underset{\sim}{\psi}\mathring{K} \rightsquigarrow \gamma \circ \psi = \underset{\sim}{\gamma} \circ \underset{\sim}{\psi}\mathring{K}$. The kernel of the later action is denoted by $\underset{\sim}{Z}$. Define the Möbius group G to be the factor group $\underset{\sim}{\Gamma}/\underset{\sim}{Z}$. The quadric Ψ is J-Hermitian: for the element $\psi = \underset{\sim}{\psi}\mathring{K}$ of Ψ, we have ${}^J\underset{\sim}{\psi} = \underset{\sim}{\psi}$. As in the classical case, ψ factors as $\psi = \bar{s}\,{}^J\bar{s}\mathring{K}$ for some $\bar{x}$ in $\bar{S}$. This induces a projective spinorial action of the Möbius group G on S given by $s = \bar{s}\Gamma \rightsquigarrow gs = \gamma\bar{s}\Gamma$, where g is the $\underset{\sim}{Z}$-coset of γ.

We will select a subset $\underset{\sim}{\Sigma}$ of the Clifford group $\underset{\sim}{\Gamma}$, an element $\varepsilon = \underset{\sim}{\varepsilon}\mathring{K}$ of the quadric ψ, and a spinor $e = \bar{e}\Gamma$ of S such that:

1. ψ is an orbit of G: $\Psi = G \circ \varepsilon = \underset{\sim}{\Sigma} \circ \varepsilon$.
2. S is an orbit of G: $S = Ge = \underset{\sim}{\Sigma}e$.
3. Each element $\underset{\sim}{\sigma}$ of $\underset{\sim}{\Sigma}$ uniquely represents a "sphere" of X.

As in the classical case, the quadric Ψ is the conformal compactification of X. It consists of copies of three structures associated with X: the vector space X itself, the cone of all isotropic vectors in X, and the projective quadric associated to this cone. We call these cases (x), (y), and (z), respectively.

Since we use these copies in the proofs to follow, we will be more precise. Let Y denote the cone $\{\mathbf{x} \text{ in } X \mid \mathbf{x}\cdot\mathbf{x} = 0\}$ and let Z denote the quadric $\{\mathbf{x}\mathring{K} \mid \mathbf{x}\cdot\mathbf{x} = 0 \text{ and } \mathbf{x} \neq 0\}$. Then, the disjoint union $X \mathbin{\mathring{\cup}} Y \mathbin{\mathring{\cup}} Z$, the set of representatives $\underset{\sim}{\Sigma}$, the set of spinors S, and the quadric Ψ correspond bijectively. The bijections $\underset{\sim}{\Sigma} \to S \to \Psi$ are given by $\underset{\sim}{\sigma} \rightsquigarrow \underset{\sim}{\sigma} \circ \varepsilon \rightsquigarrow \underset{\sim}{\sigma}e$. The bijection $X \mathbin{\mathring{\cup}} Y \mathbin{\mathring{\cup}} Z \to \underset{\sim}{\Sigma}$ consists of the following cases:

(x) For each $\mathbf{x}$ in X,

$$\mathbf{x} \rightsquigarrow \underset{\sim}{\sigma} = \begin{pmatrix} \mathbf{x} & 1 - \mathbf{x}\cdot\mathbf{x} \\ 1 & -\mathbf{x} \end{pmatrix}$$

(y) For each isotropic vector $\mathbf{y}$ in Y,

$$\mathbf{y} \rightsquigarrow \underset{\sim}{\sigma} = \begin{pmatrix} 1 & -\mathbf{y} \\ \mathbf{y} & 1 \end{pmatrix}$$

(z) For each $\mathbf{z}'\overset{\circ}{K}$ in Z, choose a representative nonzero isotropic vector $\mathbf{z}$ and a vector $\mathbf{t}$ of X such that $\mathbf{z}'\overset{\circ}{K} = \mathbf{z}\overset{\circ}{K}$, $2\mathbf{t}\cdot\mathbf{z} = 1$, and $\mathbf{t}\cdot\mathbf{t} = 1$; then

$$\mathbf{z}'K \rightsquigarrow \underset{\sim}{\sigma} = \begin{pmatrix} \mathbf{z} & 1 \\ \mathbf{zt} & -\mathbf{t} \end{pmatrix}$$

Finally, we want to call attention to an unusual feature of this paper. The method used is "form invariant" with respect to the field, the dimension of the space X, and the quadratic form associated with X. For instance, the matrix representing inversion in the proper sphere with the equation $w^0\mathbf{x}\cdot\mathbf{x} - 2\mathbf{w}\cdot\mathbf{x} + w^\infty = 0$, where $w^0 \neq 0$ and $-w^0w^\infty + \mathbf{w}\cdot\mathbf{w} \neq 0$, has the form

$$\underset{\sim}{w} = \begin{pmatrix} \mathbf{w} & -w^\infty \\ w^0 & -\mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{c} & -p \\ 1 & -\mathbf{c} \end{pmatrix} w^0$$

where $\mathbf{c}$ is the center of the sphere, and p is the power of the origin with respect to the sphere. This form does not depend on any of the above-mentioned structures of the space X. This is because, up to a nonzero scalar multiple, the elements of the matrices have direct geometrical significance. The same is true of the elements in the columns, which represent projective spinors in homogeneous coordinates. This happens because the algebraic structures which we use are directly tied to the underlying geometry.

2. PROOFS

Since the homogeneous coordinates of the set of spinors S are products of vectors in X, we begin by proving some lemmas concerned with such products. Let K be a commutative field of characteristic not two, and X be a finite-dimensional vector space over K equipped with a nonsingular quadratic form. Let A be the Clifford algebra of X, and M be the "Clifford monoid" formed by products of vectors, possibly isotropic, of X. The following lemma and its corollary are useful in showing that certain elements of the algebra A are in the monoid M.

Lemma 0. If $\mathbf{a}$ and $\mathbf{b}$ are vectors in X and l a scalar in K, then $l + \mathbf{ab}$ is in the monoid M.

Proof:

1. If $l = 0$, there is nothing to prove. If $\mathbf{a}$ is invertible, then

$$l + \mathbf{ab} = \mathbf{a}(\mathbf{a}^{-1}l + \mathbf{b})$$

is a product of two vectors, and similarly if $\mathbf{b}$ is invertible. We can now assume that $l \neq 0$ and that both $\mathbf{a}$ and $\mathbf{b}$ are isotropic.

2. If $2\mathbf{a}\cdot\mathbf{b} \neq 0$, then the vector $\mathbf{x} = \mathbf{a} + \mathbf{b}$ is invertible. The product $(l + \mathbf{ab})\mathbf{x}$ is easily checked to be a vector using $\mathbf{aba} = \mathbf{a}(2\mathbf{a}\cdot\mathbf{b}) - \mathbf{b}(2\mathbf{a}\cdot\mathbf{a})$. Since $\mathbf{x}^{-1}$ is a vector, we obtain $l + \mathbf{ab}$ is in M.

3. Suppose that $2\mathbf{a}\cdot\mathbf{b} = 0$. Since X is nonsingular, we may find isotropic vectors $\mathbf{a}'$ and $\mathbf{b}'$ such that $2\mathbf{a}\cdot\mathbf{a}' = 1$, $2\mathbf{b}\cdot\mathbf{b}' = 1$, and the nonsingular subspaces spanned by $\mathbf{a}$ and $\mathbf{a}'$ and by $\mathbf{b}$ and $\mathbf{b}'$ are orthogonal. Choose k in $\mathring{K}$ not equal to ± 1, so that the vectors $\mathbf{y} = \mathbf{a} + \mathbf{a}'k$ and $\mathbf{z} = \mathbf{a} - \mathbf{a}'k$ are nonisotropic. One checks that the product $\mathbf{y}(l + \mathbf{ab})\mathbf{z}$ is equal to $L + \mathbf{AB}$, where again $L = kl$ is a nonzero scalar and both $\mathbf{A} = 2\mathbf{a}l + \mathbf{b}k$ and $\mathbf{B} = -\mathbf{a}'k$ are isotropic vectors. Moreover, $2\mathbf{A}\cdot\mathbf{B} = -2kl \neq 0$. This is of the form covered by part 2. Since $\mathbf{y}$ and $\mathbf{z}$ are invertible, it follows that $l + \mathbf{ab}$ is in M. ■

Corollary. If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in X, then both $\mathbf{ab} + \mathbf{bc}$ and $\mathbf{a} + \mathbf{bac}$ are in the monoid M.

Remark. That is, if both terms have a vector in common, their sum is in the monoid.

Proof. Since $\mathbf{ab} + \mathbf{bc} = 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}(\mathbf{c} - \mathbf{a})$, it follows from the lemma that $\mathbf{ab} + \mathbf{bc}$ is in M. We prove that $\mathbf{a} + \mathbf{bac}$ is in M by considering cases.

1. Suppose that $\mathbf{a}\cdot\mathbf{a} \neq 0$. Since $\mathbf{a}^{-1}\mathbf{ba}$ is a vector, $\mathbf{a} + \mathbf{bac} = \mathbf{a}(1 + \mathbf{a}^{-1}\mathbf{bac})$ is in M by the lemma.

2. Suppose that $2\mathbf{a}\cdot\mathbf{c} = 0$. Then $\mathbf{a} + \mathbf{bac} = (1 - \mathbf{bc})\mathbf{a}$, which is in M by the lemma.

3. The remaining case is that both $\mathbf{a}\cdot\mathbf{a} = 0$ and $2\mathbf{a}\cdot\mathbf{c} \neq 0$. Choose k so that $\mathbf{x} = \mathbf{a}k + \mathbf{c}$ is an invertible vector. Then

$$(\mathbf{a} + \mathbf{bac})\mathbf{x} = \mathbf{ac} + \mathbf{b}(2\mathbf{a}\cdot\mathbf{c}k + \mathbf{c}\cdot\mathbf{c})\mathbf{a}$$

and this is in M by the first assertion of this corollary. Since $\mathbf{x}$ is invertible, $\mathbf{a} + \mathbf{bac}$ is in M also. ■

Lemma 1. If X is nonsingular, then all the nonzero elements of the Clifford monoid are of the form $m = \mathbf{z}_1\mathbf{z}_2 \ldots \mathbf{z}_r\gamma$, where the $\mathbf{z}_i$ are pairwise nonzero orthogonal isotropic vectors and γ is an element of Γ.

Call such an expression a "left-reduced" form of the element m.

Proof. View m as the product of elements in order from three "lists": $m = (\ldots \mathbf{z} \ldots)(\ldots \mathbf{x} \ldots)(\ldots \mathbf{t} \ldots)$. The t-list consists of nonisotropic vectors, and the z-list consists of isotropic vectors which are orthogonal to all vectors in both the z-list and the x-list. Initially the x-list is any product of vectors giving the element m. On the other hand, initially both the z- and the t-lists are empty, that is, the product of each by convention is the scalar 1. Each step of the induction will examine the rightmost vector $\mathbf{x}'$ of the x-list and then reduce by one the number of vectors in that list and increase by one the size of either the z- or the t-list.

1. If $\mathbf{x}'$ is nonisotropic, then simply declare $\mathbf{x}'$ to be the leftmost member of the t-list with no other changes.

2. Else $\mathbf{x}'$ is isotropic.

(a) If $\mathbf{x}'$ is orthogonal to every vector in the x-list, then move $\mathbf{x}'$ to the right of the z-list with at most a change of sign.

(b) Else let $\mathbf{x}''$ be the first vector to the left of $\mathbf{x}'$ in the x-list which is not orthogonal to $\mathbf{x}'$. Move $\mathbf{x}'$ to the left until it is adjacent to $\mathbf{x}''$ with at most a change of sign.

(i) If $\mathbf{x}''$ is nonisotropic, then shift $\mathbf{x}''$ to the t-list by conjugating the vectors in the x-list to its right.

(ii) Else $\mathbf{x}''$ is also isotropic. Then replace $\mathbf{x}'$ by the nonisotropic vector $\mathbf{x}''+\mathbf{x}'$. Now shift this nonisotropic vector to the t-list by conjugating the vectors in the x-list to its right.

The proof is now complete with the observation that if any vector $\mathbf{z}$ and any nonisotropic vector $\mathbf{t}$ are both orthogonal to any other vector $\mathbf{z}'$, then the conjugate $\mathbf{tzt}^{-1}$ remains orthogonal to $\mathbf{z}'$. ■

Remark. All the nonzero elements of the Clifford monoid M are also of the form $m=\gamma\mathbf{z}_r\ldots\mathbf{z}_2\mathbf{z}_1$, where the $\mathbf{z}_i$ are pairwise nonzero orthogonal isotropic vectors and γ is an element of Γ. Call such an expression a "right-reduced" form of the element m. This is immediate from the proof of the lemma by interchanging right and left.

Lemma 2 and its corollaries are used to determine the special form for spinors in the (z)-case. This case has no classical analogue to guide us.

Definition. Define, within the Clifford algebra, the exterior product of vectors in X to be

$$\mathbf{x}_1\wedge\mathbf{x}_2\wedge\cdots\wedge\mathbf{x}_r=\frac{1}{r!}\sum_{i\in S_r}\mathbf{x}_{i(1)}\mathbf{x}_{i(2)}\ldots\mathbf{x}_{i(r)}\operatorname{sgn}(i)$$

where $\operatorname{sgn}(i)$ is the sign of the permutation i in the symmetric group S_r on the integers from 1 to r.

Remark. The vectors $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_r$ are linearly independent if and only if $\mathbf{x}_1\wedge\mathbf{x}_2\wedge\cdots\wedge\mathbf{x}_r\neq 0$.

Notation as in Lemma 1.

Lemma 2. If $\mathbf{z}_1\mathbf{z}_2\ldots\mathbf{z}_r\gamma$ and $\mathbf{z}'_1\mathbf{z}'_2\ldots\mathbf{z}'_{r'}\gamma'$ are two left-reduced forms of the same nonzero element of M, then $r=r'$ and the z-list and the z'-list span the same subspace of X.

Proof. Suppose that some $\mathbf{z}'_j$ is not in the span of $\mathbf{z}_1,\mathbf{z}_2,\ldots,\mathbf{z}_r$. Then, since the exterior and Clifford products are the same for pairwise orthogonal vectors, we have

$$\begin{aligned}0 &\neq (\mathbf{z}_j' \wedge \mathbf{z}_1 \wedge \mathbf{z}_2 \wedge \cdots \wedge \mathbf{z}_r)\gamma \\ &= \mathbf{z}_j' \wedge (\mathbf{z}_1\mathbf{z}_2 \ldots \mathbf{z}_r\gamma) \\ &= \mathbf{z}_j' \wedge (\mathbf{z}_1'\mathbf{z}_2' \ldots \mathbf{z}_{r'}'\gamma') \\ &= (\mathbf{z}_j' \wedge \mathbf{z}_1' \wedge \mathbf{z}_2' \wedge \cdots \wedge \mathbf{z}_{r'}')\gamma' = 0\end{aligned}$$

which is a contradiction. Thus, all the $\mathbf{z}_j'$ are in the span of $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_r$ and $r' \leq r$. Now repeat the argument with the z-list and z'-list interchanged. ■

Corollary 1. If $\mathbf{z}\gamma$ and $\mathbf{z}'\gamma'$ are left reduced forms of the same nonzero element of M, then $\mathbf{z}' = \mathbf{z}k$ for some scalar $k \neq 0$.

Remarks. Call the size of an element the size of its z-list in any left- or right-reduced form. The size of a product mn is greater than or equal to the maximum of the sizes of m and n. This follows since the steps in the reduction process in the proof of Lemma 1, and the remark following it, never decrease the number of elements in a z-list. In particular, if two elements of M which are not invertible have a product of size one, then each element must have size one.

Recall

$$\bar{S} = \left\{ \bar{s} = \begin{pmatrix} s^1 \\ s^0 \end{pmatrix} \in \bar{A} \,\middle|\, s^1 \text{ and } s^0 \text{ in } M,\ s^1 {}^*s^0 \text{ is in } X, \right.$$
$$\left. \text{but if neither } s^1 \text{ nor } s^0 \text{ is in } \Gamma, \text{ then } s^1 {}^*s^0 \neq 0 \right\}$$

and $S = \{s = \bar{s}\Gamma \mid \bar{s} \text{ in } \bar{S}\}$.

Corollary 2. Let

$$\bar{s} = \begin{pmatrix} s^1 \\ s^0 \end{pmatrix}$$

be in $\bar{S}$, and assume that neither s^1 nor s^0 is in Γ, so that $s^1 {}^*s^0 \neq 0$ is in X, and let $\mathbf{z}$ be a nonzero isotropic vector defined up to a scalar multiple by $\mathbf{z}\mathring{K} = s^1 {}^*s^0\mathring{K}$. Then there exists a vector $\mathbf{t}$ in X so that $\mathbf{t}\cdot\mathbf{t} = 1$, $2\mathbf{t}\cdot\mathbf{z} = 1$, and

$$\bar{s} = \begin{pmatrix} \mathbf{z} \\ \mathbf{zt} \end{pmatrix} \gamma$$

where γ is in the Clifford group Γ.

Proof. Define $\mathbf{z} = s^1 {}^*s^0 k$, where k in $\mathring{K}$ is to be chosen later. Since $\mathbf{z} \neq 0$, Corollary 1 and the subsequent remarks imply that s^1 and ${}^*s^0$ have left- and right-reduced forms $s^1 = \mathbf{z}\gamma$ and $^*s^0 = \delta\mathbf{z}$, respectively. Thus, $\mathbf{z} = (\mathbf{z}\gamma)(\delta\mathbf{z})k = \mathbf{z}\mathbf{z}'\gamma\delta k$, where $\mathbf{z}' = (\gamma\delta)\mathbf{z}(\gamma\delta)^{-1}$. Note that $\mathbf{z}$ cannot be orthogonal to $\mathbf{z}'$, for if it were, then $\mathbf{z}\mathbf{z}'$ would be zero or a z-list of size two instead of one. Set $\mathbf{t}' = \mathbf{z}(2\mathbf{z}'\cdot\mathbf{z}) + \mathbf{z}'$, so now $\mathbf{z} = \mathbf{z}\mathbf{t}'\gamma\delta k$. Thus,

$$s^0 = {}^*(\delta\mathbf{z}) = -\mathbf{z}\,{}^*\delta = -\mathbf{z}\mathbf{t}'\gamma\delta\,{}^*\delta k = \mathbf{z}(-\mathbf{t}'\delta\,{}^*\delta k)\gamma = \mathbf{z}\mathbf{t}\gamma$$

where $\mathbf{t} = -\mathbf{t}'\delta\,{}^*\delta k$. It follows that

$$\bar{s} = \begin{pmatrix} \mathbf{z} \\ \mathbf{z}\mathbf{t} \end{pmatrix} \gamma$$

and that

$$2\mathbf{t}\cdot\mathbf{z} = -2\mathbf{t}'\cdot\mathbf{z}\delta\,{}^*\delta k = -(2\mathbf{z}'\cdot\mathbf{z})\delta\,{}^*\delta k \neq 0$$

Choose $k = -1/(2\mathbf{z}'\cdot\mathbf{z})\delta\,{}^*\delta$ to achieve $2\mathbf{t}\cdot\mathbf{z} = 1$ and $\mathbf{t}\cdot\mathbf{t} = 1$. ∎

Recall that there are three cases, called (x), (y), and (z), which describe points of the conformal compactification of X.

Also recall that the anti-involution J is given by $\underset{\sim}{a} \rightsquigarrow {}^J\underset{\sim}{a} = \underset{\sim}{\theta}\,{}^{t*}\underset{\sim}{a}\underset{\sim}{\theta}^{-1}$ and induces the "Hermitian adjoint" j given by $\bar{a} \rightsquigarrow {}^j\bar{a} = {}^{t*}\bar{a}\underset{\sim}{\theta}^{-1}$, where

$$\underset{\sim}{\theta} = \begin{pmatrix} \mathbf{0} & 1 \\ 1 & \mathbf{0} \end{pmatrix}$$

$*$ acts on the entries of a matrix elementwise, and t is the ordinary transpose, so that ${}^j(\underset{\sim}{\gamma}\bar{a}) = {}^j\bar{a}\,{}^J\underset{\sim}{\gamma}$, for all $\underset{\sim}{\gamma}$ in $\underset{\sim}{\Gamma}$.

Notation as before. In addition, from Section 1,

$$\underset{\sim}{W} = \left\{ \underset{\sim}{w} = \begin{pmatrix} \mathbf{w} & -w^\infty \\ w^0 & -\mathbf{w} \end{pmatrix} \middle|\, w^0 \text{ and } w^\infty \text{ are scalars in } K \text{ and } \mathbf{w} \text{ is a vector in } X \right\}$$

$$\underset{\sim}{\Psi} = \{\underset{\sim}{w} \text{ in } \underset{\sim}{W} \,|\, \underset{\sim}{w} \text{ not zero, and } \underset{\sim}{w}\cdot\underset{\sim}{w} = 0\}$$

$$\Psi = \{\psi = \underset{\sim}{\psi}\mathring{K} \,|\, \underset{\sim}{\psi} \text{ in } \underset{\sim}{\Psi}\}$$

Let $\bar{e}$ be an element of $\bar{S}$ and set $e = \bar{e}\Gamma$ in S. One checks that $\underset{\sim}{\varepsilon} = \bar{e}\,{}^j\bar{e}$ is in $\underset{\sim}{\Psi}$, so $\varepsilon = \underset{\sim}{\varepsilon}\mathring{K}$ is in Ψ. Note that $\underset{\sim}{\sigma} \circ \varepsilon = \underset{\sim}{\sigma}\underset{\sim}{\varepsilon}\,{}^J\underset{\sim}{\sigma}\mathring{K}$ equals $\underset{\sim}{\sigma}\bar{e}\,{}^j(\underset{\sim}{\sigma}\bar{e})\mathring{K}$. We may write $S = \underset{\sim}{\Sigma}e$ and $\Psi = \underset{\sim}{\Sigma} \circ \varepsilon$ as a consequence of the following lemma.

Lemma 3. There is a choice of $\bar{e}$ in $\bar{S}$ and a subset $\underset{\sim}{\Sigma}$ of $\underset{\sim}{\Gamma}$ which is a set of unique representatives of both S and Ψ in the sense that the maps $\underset{\sim}{\sigma} \rightsquigarrow \underset{\sim}{\sigma}e$ and $\underset{\sim}{\sigma} \rightsquigarrow \underset{\sim}{\sigma} \circ \varepsilon$ are bijections from $\underset{\sim}{\Sigma}$ to S and Ψ, respectively. Hence $\underset{\sim}{\sigma}e \rightsquigarrow \underset{\sim}{\sigma} \circ \varepsilon$ is a bijection from S to Ψ.

Proof. A typical element of S is

$$s = \bar{s}\Gamma, \qquad \bar{s} = \begin{pmatrix} s^1 \\ s^0 \end{pmatrix}$$

where s^1 and s^0 are elements of M such that $s^1 * s^0$ is a vector of X, but if neither s^1 nor s^0 is in Γ, then $s^1 * s^0 \neq 0$. A typical element of Ψ is $\psi = \underset{\sim}{\psi}\mathring{K}$,

$$\underset{\sim}{\psi} = \begin{pmatrix} \boldsymbol{\psi} & -\psi^\infty \\ \psi^0 & -\boldsymbol{\psi} \end{pmatrix}$$

where ψ^0 and ψ^∞ are scalars in K and $\boldsymbol{\psi}$ is a vector in X, such that $\underset{\sim}{\psi} \neq \underset{\sim}{0}$ and $\underset{\sim}{\psi} \cdot \underset{\sim}{\psi} = -\psi^0\psi^\infty + \boldsymbol{\psi}\cdot\boldsymbol{\psi} = 0$. Choose $\bar{e}$ to be $\binom{1}{0}$, and check that e is in S.

A unique representative $\underset{\sim}{\sigma}$ will be constructed that represents both an s in S and the corresponding ψ in Ψ. For the construction of $\underset{\sim}{\Sigma}$, we consider separately the cases (x), (y), and (z) of elements in Ψ, the conformal compactification of X.

(x) Case. This is the case when s^0 is in Γ and, correspondingly, ψ^0 is in $\mathring{K}$. For each $s = \binom{s^1}{s^0}\Gamma$ in S such that s^0 is in Γ, dependent on s but not on the choice of $\binom{s^1}{s^0}$, we define $\mathbf{x}$ to be the vector $s^1(s^0)^{-1} = s^1 * s^0 / s^0 * s^0$ of X. Then $s = \binom{\mathbf{x}}{1}\Gamma$. The corresponding ψ is defined by

$$\Psi = \begin{pmatrix} \mathbf{x} & -\mathbf{x}\cdot\mathbf{x} \\ 1 & -\mathbf{x} \end{pmatrix} \mathring{K}$$

Then ψ is in Ψ and for any $\underset{\sim}{\psi}$ such that $\psi = \underset{\sim}{\psi}\mathring{K}$, ψ^0 is in $\mathring{K}$. Conversely, for each

$$\psi = \begin{pmatrix} \boldsymbol{\psi} & -\psi^\infty \\ \psi^0 & -\boldsymbol{\psi} \end{pmatrix} \mathring{K}$$

in Ψ such that ψ^0 is in $\mathring{K}$, dependent on ψ but not on the choice of

$$\underset{\sim}{\psi} = \begin{pmatrix} \boldsymbol{\psi} & -\psi^\infty \\ \psi^0 & -\boldsymbol{\psi} \end{pmatrix}$$

we define $\mathbf{x}$ to be the vector $\boldsymbol{\psi}(\psi^0)^{-1}$ of X. Then

$$\psi = \begin{pmatrix} \mathbf{x} & -\mathbf{x}\cdot\mathbf{x} \\ 1 & -\mathbf{x} \end{pmatrix} \mathring{K}$$

Any $\mathbf{x}$ may be obtained for a suitable choice of ψ. This means that X is embedded in its conformal compactification Ψ by identifying vectors of X with vertices of cones in Ψ. The corresponding s is obtained by setting $s = \binom{\mathbf{x}}{1}\Gamma$. Then s is in S, and for any $\bar{s}$ such that $s = \bar{s}\Gamma$ still s^0 is in Γ. Having found the corresponding pair, we now define $\underset{\sim}{\sigma}$, their common representative, to be the invertible vector

$$\begin{pmatrix} \mathbf{x} & 1-\mathbf{x}\cdot\mathbf{x} \\ 1 & -\mathbf{x} \end{pmatrix}$$

of $\underset{\sim}{W}$. One easily checks that $s = \underset{\sim}{\sigma}e$ and $\psi = \underset{\sim}{\sigma} \circ \varepsilon$. For each $\mathbf{x}$ in X we include

$$\underset{\sim}{\sigma} = \begin{pmatrix} \mathbf{x} & 1-\mathbf{x}\cdot\mathbf{x} \\ 1 & -\mathbf{x} \end{pmatrix}$$

into $\underset{\sim}{\Sigma}$ as the unique representative of both s and ψ.

(y) Case. This is the case when s^1 is in Γ but s^0 is not and, correspondingly, ψ^∞ is in $\mathring{K}$ but ψ^0 is not. For each $s = \binom{s^1}{s^0}\Gamma$ in S such that s^0 is not in Γ but s^1 is in Γ, dependent on s but not on the choice of $\binom{s^1}{s^0}$, we define $\mathbf{y}$ to be the isotropic vector $s^0(s^1)^{-1} = s^0 * s^1/s^1 * s^1$ of X. Then $s = \binom{1}{\mathbf{y}}\Gamma$. Define the corresponding ψ by

$$\psi = \begin{pmatrix} -\mathbf{y} & 1 \\ 0 & \mathbf{y} \end{pmatrix} \mathring{K}$$

Then ψ is in Ψ and ψ^0 is not in $\mathring{K}$, but ψ^∞ is in $\mathring{K}$ for any $\underset{\sim}{\psi}$ such that $\psi = \underset{\sim}{\psi}\mathring{K}$. Conversely, for each

$$\psi = \begin{pmatrix} \boldsymbol{\psi} & -\psi^\infty \\ \psi^0 & -\boldsymbol{\psi} \end{pmatrix} \mathring{K}$$

in Ψ such that ψ^0 is not in $\mathring{K}$, that is, $\psi^0 = 0$, but ψ^∞ is in $\mathring{K}$, dependent on ψ but not on the choice of

$$\begin{pmatrix} \boldsymbol{\psi} & -\psi^\infty \\ \psi^0 & -\boldsymbol{\psi} \end{pmatrix}$$

we define $\mathbf{y}$ to be the isotropic vector $\boldsymbol{\psi}(\psi^\infty)^{-1}$ of X. Then

$$\psi = \begin{pmatrix} -\mathbf{y} & 1 \\ 0 & \mathbf{y} \end{pmatrix} \mathring{K}$$

The corresponding s is obtained by setting $s = \binom{1}{\mathbf{y}}\Gamma$. Then s is in S, and for any $\bar{s}$ such that $s = \bar{s}\Gamma$, we have s^0 is not in Γ and s^1 is in Γ. Define $\underset{\sim}{\sigma}$, their common representative, to be

$$\begin{pmatrix} 1 & -\mathbf{y} \\ \mathbf{y} & 1 \end{pmatrix} = \begin{pmatrix} -\mathbf{y} & 1 \\ 1 & \mathbf{y} \end{pmatrix} \begin{pmatrix} \mathbf{0} & 1 \\ 1 & \mathbf{0} \end{pmatrix}$$

which is a product of two invertible vectors of $\underset{\sim}{W}$, showing that $\underset{\sim}{\sigma}$ is in $\underset{\sim}{\Gamma}$. One again easily checks that $s = \underset{\sim}{\sigma} e$ and $\psi = \underset{\sim}{\sigma} \circ \varepsilon$. For each isotropic vector $\mathbf{y}$ in X, we include

$$\underset{\sim}{\sigma} = \begin{pmatrix} 1 & -\mathbf{y} \\ \mathbf{y} & 1 \end{pmatrix}$$

into $\underset{\sim}{\Sigma}$ as the unique representative of both s and ψ.

(z) Case. This is the case when neither s^0 nor s^1 is in Γ and, correspondingly, neither ψ^0 nor ψ^∞ is in $\mathring{K}$. In this case, $s^1 {}^* s^0$ is a nonzero isotropic vector in X. No unique vector of X is determined by s, but $\mathbf{z}$ is determined up to a nonzero scalar multiple by $\mathbf{z}\mathring{K} = s^1 {}^* s^0 \mathring{K}$ for any choice of $\binom{s^1}{s^0}$. Then

$$s = \begin{pmatrix} \mathbf{z} \\ \mathbf{zt} \end{pmatrix} \Gamma$$

by the corollary to Lemma 2, and where $\mathbf{t}$ is any vector of X such that $2\mathbf{t}\cdot\mathbf{z} = 1$ and $\mathbf{t}\cdot\mathbf{t} = 1$. The corresponding ψ is obtained by setting

$$\psi = \begin{pmatrix} \mathbf{z} & 0 \\ 0 & -\mathbf{z} \end{pmatrix} \mathring{K}$$

Then ψ is in Ψ, and for any $\underset{\sim}{\psi}$ such that $\psi = \underset{\sim}{\psi}\mathring{K}$ both ψ^0 and ψ^∞ are not in $\mathring{K}$. Conversely, for each

$$\psi = \begin{pmatrix} \boldsymbol{\psi} & -\psi^\infty \\ \psi^0 & -\boldsymbol{\psi} \end{pmatrix} \mathring{K}$$

in Ψ such that both ψ^0 and ψ^∞ are not in $\mathring{K}$, again no unique vector of X is determined by ψ, but $\mathbf{z}$ is determined up a nonzero scalar multiple by $\mathbf{z}\mathring{K} = \boldsymbol{\psi}\mathring{K}$ independent of

$$\begin{pmatrix} \boldsymbol{\psi} & -\psi^\infty \\ \psi^0 & -\boldsymbol{\psi} \end{pmatrix}$$

Then

$$\psi = \begin{pmatrix} \mathbf{z} & 0 \\ 0 & -\mathbf{z} \end{pmatrix} \mathring{K}$$

The corresponding s is obtained by setting

$$s = \begin{pmatrix} \mathbf{z}' \\ \mathbf{z}'\mathbf{t}' \end{pmatrix} \Gamma$$

where $\mathbf{z}' = \mathbf{z}k$ is a nonzero isotropic vector and $\mathbf{t}'$ is any vector of X such that $2\mathbf{t}' \cdot \mathbf{z}' \neq 0$. We note that we may omit the primes, for if $\mathbf{z}' = \mathbf{z}k$, k in $\mathring{K}$, then

$$\begin{pmatrix} \mathbf{z}' \\ \mathbf{z}'\mathbf{t}' \end{pmatrix} \Gamma = \begin{pmatrix} \mathbf{z} \\ \mathbf{zt} \end{pmatrix} \Gamma$$

Since

$$\begin{pmatrix} \mathbf{z}' \\ \mathbf{z}'\mathbf{t}' \end{pmatrix} = \begin{pmatrix} \mathbf{z} \\ \mathbf{z}\mathbf{t} \end{pmatrix} (\mathbf{t}\mathbf{z}+\mathbf{z}\mathbf{t}')k$$

setting $\gamma = (\mathbf{tz}+\mathbf{zt}')k$, we have $\gamma^*\gamma = (2\mathbf{t}'\cdot\mathbf{z})k^2 \neq 0$ and γ is in Γ by the corollary of Lemma 1. Now s is in S and both s^0 and s^1 are not in Γ. Now define $\underset{\sim}{\sigma}$, the common representative, to be

$$\underset{\sim}{\sigma} = \begin{pmatrix} \mathbf{z} & 1 \\ \mathbf{zt} & -\mathbf{t} \end{pmatrix} = \begin{pmatrix} \mathbf{v}l & ml \\ \mathbf{v}\cdot\mathbf{v} & -\mathbf{v}l \end{pmatrix} \begin{pmatrix} \mathbf{v} & m \\ 0 & -\mathbf{v} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{z} & 1 \\ l & -\mathbf{z} \end{pmatrix}^{-1}$$

where $m = 1 - l$ is a scalar, $\mathbf{v} = \mathbf{z} - \mathbf{t}l$ is a vector, and l is a nonzero scalar chosen so $\mathbf{v}$ is nonisotropic. This makes $\underset{\sim}{\sigma}$ a product of three invertible vectors of $\underset{\sim}{W}$. Again one easily checks that $s = \underset{\sim}{\sigma}e$ and $\psi = \underset{\sim}{\sigma} \circ \varepsilon$. Thus, for each $\mathbf{z}'\mathring{K}$, we choose a representative nonzero isotropic vector $\mathbf{z}$ and a vector $\mathbf{t}$ such that $2\mathbf{t}\cdot\mathbf{z} = 1$ and $\mathbf{t}\cdot\mathbf{t} = 1$ and include

$$\underset{\sim}{\sigma} = \begin{pmatrix} \mathbf{z} & 1 \\ \mathbf{zt} & -\mathbf{t} \end{pmatrix}$$

in $\underset{\sim}{\Sigma}$ as the unique representative of s and ψ.

Since we have constructed a set of unique representatives $\underset{\sim}{\sigma}$ in $\underset{\sim}{\Sigma}$ for each corresponding pair $s = \underset{\sim}{\sigma}e$ and $\psi = \underset{\sim}{\sigma} \circ \varepsilon$, clearly $\underset{\sim}{\sigma}e \rightsquigarrow \underset{\sim}{\sigma} \circ \varepsilon$ is a bijection from $S = \underset{\sim}{\Sigma}e$ to $\Psi = \underset{\sim}{\Sigma} \circ \varepsilon$. ■

To show that Ψ and S are preserved under the action of the group $\underset{\sim}{\Gamma}$, it suffices to show that they are preserved under the action of invertible matrices in $\underset{\sim}{W}$, since elements of $\underset{\sim}{\Gamma}$ are products of invertible matrices in $\underset{\sim}{W}$.

Notation as in Lemma 3.

Lemma 4. $(\underset{\sim}{W} \cap \underset{\sim}{\Gamma}) \circ \Psi \subseteq \Psi$.

Proof. A typical element of $\underset{\sim}{W} \cap \underset{\sim}{\Gamma}$ is an invertible vector

$$\underset{\sim}{w} = \begin{pmatrix} \mathbf{w} & -w^{\infty} \\ w^0 & -\mathbf{w} \end{pmatrix}$$

of $\underset{\sim}{W}$, and a typical element of $\Psi = \underset{\sim}{\Sigma} \circ \varepsilon$ is $\psi = \underset{\sim}{\sigma} \circ \varepsilon = \underset{\sim}{\psi}\mathring{K}$. Let $\underset{\sim}{\psi}' = \underset{\sim}{w}\underset{\sim}{\psi}^J\underset{\sim}{w}$ and $\psi' = \underset{\sim}{\psi}'\mathring{K} = \underset{\sim}{w} \circ \psi$. Now we must show that ψ' is still Ψ. But

$$\underset{\sim}{\psi}' = \underset{\sim}{w}\underset{\sim}{\psi}^J\underset{\sim}{w} = \underset{\sim}{w}\underset{\sim}{\psi}\underset{\sim}{w} = \underset{\sim}{w}(2\underset{\sim}{w}\cdot\underset{\sim}{\psi}) - \underset{\sim}{\psi}(\underset{\sim}{w}\cdot\underset{\sim}{w})$$

is in $\tilde{W}$. In addition, $\underset{\sim}{\psi}'$ is isotropic since $\underset{\sim}{\psi}'\cdot\underset{\sim}{\psi}' = (\underset{\sim}{w}\cdot\underset{\sim}{w})^2\underset{\sim}{\psi}\cdot\underset{\sim}{\psi} = 0$. Finally, $\underset{\sim}{\psi}'$ is not zero, since $\underset{\sim}{\psi}$ is not zero and $\underset{\sim}{w}$ is invertible. Thus, ψ' is in Ψ. ■

Corollary. $(\underset{\sim}{W} \cap \underset{\sim}{\Gamma})S \subseteq S$.

Proof. A typical element of $\underset{\sim}{W} \cap \underset{\sim}{\Gamma}$ is an invertible vector

$$\underset{\sim}{w} = \begin{pmatrix} \mathbf{w} & -w^{\infty} \\ w^0 & -\mathbf{w} \end{pmatrix}$$

of $\underset{\sim}{W}$, and a typical element of $S = \underset{\sim}{\Sigma} e$ is of the form

$$\underset{\sim}{\sigma} \bar{e} \Gamma = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \Gamma, \quad \begin{pmatrix} 1 \\ \mathbf{y} \end{pmatrix} \Gamma, \quad \text{or} \quad \begin{pmatrix} \mathbf{z} \\ \mathbf{zt} \end{pmatrix} \Gamma$$

as in Lemma 2. Their product is of the form $\bar{a}\Gamma$, where $\bar{a} = \binom{a^1}{a^0}$. Using Lemma 0 and its corollary, in all cases one sees that a^1 and a^0 are in M. It is also easy to calculate that $a^1 * a^0$ is a vector in X. Finally, we must show that, if $a^1 * a^0 = 0$, then either a^1 or a^0 is invertible. For this, note that

$$\bar{a}\,{}^j\bar{a} = \begin{pmatrix} a^1 \\ a^0 \end{pmatrix} ({}^*a^0 \; {}^*a^1) = \begin{pmatrix} a^1 * a^0 & a^1 * a^1 \\ a^0 * a^0 & a^0 * a^1 \end{pmatrix}$$

as well as $\bar{a}\,{}^j\bar{a} = (\underset{\sim}{w}\underset{\sim}{\sigma}\bar{e})^j(\underset{\sim}{w}\underset{\sim}{\sigma}\bar{e}) = \underset{\sim}{w}\underset{\sim}{\psi}\,{}^J\underset{\sim}{w}$, where $\underset{\sim}{\psi} = \underset{\sim}{\sigma}\bar{e}\,{}^j(\underset{\sim}{\sigma}\bar{e})$. Here we have used ${}^j(\underset{\sim}{w}\underset{\sim}{\sigma}\bar{e}) = {}^j(\underset{\sim}{\sigma}\bar{e})^J\underset{\sim}{w}$. By the lemma, $\underset{\sim}{w}\underset{\sim}{\psi}\,{}^J\underset{\sim}{w}\mathring{\underset{\sim}{K}}$ is in Ψ, so $\bar{a}\,{}^j\bar{a} \neq 0$. But if $a^1 * a^0 = 0$, then $a^0 * a^1 = 0$ also, and

$$\bar{a}\,{}^j\bar{a} = \begin{pmatrix} \mathbf{0} & a^1 * a^1 \\ a^0 * a^0 & \mathbf{0} \end{pmatrix}$$

So, either $a^0 * a^0$ or $a^1 * a^1$ is a nonzero scalar, showing that a^1 or a^0 is invertible. This, $\bar{a}\Gamma$ is in S. ■

The lemma to follow will be used to develop the equations associated with Möbius transformations and their fixed points. This lemma, in the context of the complex numbers, was alluded to by Cartan (1937), who remarks that "spinors have metric properties, but not affine characteristics."

Lemma 5. For $\bar{s}$ in $\bar{S}$, ${}^j\bar{s}\bar{s} = 0$.

Proof. We have $\bar{s} = \underset{\sim}{\sigma}\bar{e}$ with $\underset{\sim}{\sigma}$ in $\underset{\sim}{\Sigma}$. Then ${}^j\bar{s}\bar{s} = {}^j\bar{e}\,{}^J\underset{\sim}{\sigma}\underset{\sim}{\sigma}\bar{e} = {}^j\bar{e}\bar{e}\,{}^J\underset{\sim}{\sigma}\underset{\sim}{\sigma}$, since ${}^J\underset{\sim}{\sigma}\underset{\sim}{\sigma}$ is a scalar times the unit matrix. Since $\bar{e} = \binom{1}{0}$ and ${}^j\bar{e} = (\mathbf{0}, 1)$, we have ${}^j\bar{e}\bar{e} = 0$. ■

Corollary. Let $\underset{\sim}{\gamma}$ be in $\underset{\sim}{\Gamma}$ and $\bar{s}$ in $\bar{S}$. If $\bar{s}' = \underset{\sim}{\gamma}\bar{s}$, then ${}^j\bar{s}'\underset{\sim}{\gamma}\bar{s} = 0$.

Remarks. If we set

$$\underset{\sim}{\gamma} = \begin{pmatrix} \gamma^1 & \gamma^{\infty} \\ \gamma^0 & \gamma^2 \end{pmatrix} \qquad \text{and} \qquad \bar{s} = \begin{pmatrix} s^1 \\ s^0 \end{pmatrix}$$

then

$$\begin{aligned} {}^j\bar{s}'\underset{\sim}{\gamma}\bar{s} &= ({}^*s'^0 \; {}^*s'^1) \begin{pmatrix} \gamma^1 & \gamma^{\infty} \\ \gamma^0 & \gamma^2 \end{pmatrix} \begin{pmatrix} s^1 \\ s^0 \end{pmatrix} \\ &= {}^*s'^1\gamma^0 s^1 + {}^*s'^1\gamma^2 s^0 + {}^*s'^0\gamma^1 s^1 + {}^*s'^0\gamma^{\infty} s^0 \end{aligned}$$

from which we obtain the equation for the general Möbius transformation associated with $\underset{\sim}{\gamma}$ in its biquadratic form over the Clifford algebra A.

Recall that an element of G is $g = \underset{\sim}{\gamma}\underset{\sim}{Z}$, where $\underset{\sim}{Z}$ is the kernel of $\underset{\sim}{\Gamma}$ acting on Ψ. So G represents the Möbius transformations directly, while $\underset{\sim}{\Gamma}$ represents their equations.

The Möbius transformation $\underset{\sim}{\gamma}$ represents inversion in a nonsingular "sphere" of X, when $\underset{\sim}{\gamma}$ is

$$\underset{\sim}{w} = \begin{pmatrix} \mathbf{w} & -w^{\infty} \\ w^0 & -\mathbf{w} \end{pmatrix}$$

In this case,

$${}^j\bar{s}'\underset{\sim}{w}\bar{s} = {}^*s'^1 w^0 s^1 - {}^*s'^1 \mathbf{w} s^0 + {}^*s'^0 \mathbf{w} s^1 - {}^*s'^0 w^{\infty} s^0$$

The corresponding equation of fixed points, ${}^j\bar{s}\underset{\sim}{w}\bar{s} = 0$, represents the points of nonsingular "sphere." In the (x) case of Lemma 3, that is,

$$\bar{s} = \begin{pmatrix} s^1 \\ s^0 \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \gamma, \qquad x \text{ in } X \text{ and } \underset{\sim}{\gamma} \text{ in } \underset{\sim}{\Gamma}$$

the equation reduces to $w^0 \mathbf{x}\cdot\mathbf{x} - 2\mathbf{w}\cdot\mathbf{x} + w^{\infty} = 0$. This is the classical equation of a "sphere."

Besides inversion in a nonsingular "sphere" of X, other common Möbius transformations of X have simple associated matrices.

Translation:

$$\mathbf{x} \rightsquigarrow \mathbf{x} + \mathbf{a} \qquad \text{with} \quad \begin{pmatrix} 1 & \mathbf{a} \\ 0 & 1 \end{pmatrix}$$

Homothety:

$$\mathbf{x} \rightsquigarrow \mathbf{x}\lambda \quad (\lambda \text{ in } \mathring{K}) \qquad \text{with} \quad \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$$

"*Special conformal*" *transformation*:

$$\mathbf{x} \rightsquigarrow (\mathbf{x}^{-1} + \mathbf{a})^{-1} \quad (\text{where defined}) \qquad \text{with} \quad \begin{pmatrix} 1 & 0 \\ \mathbf{a} & 1 \end{pmatrix}$$

Compare this with the approach of Lounesto and Latvamaa (1980).

This completes the technical lemmas. We may now state our main conclusions.

Proposition 1. The set of projective spinors is $S = \underset{\sim}{\Gamma} e$, and the Möbius quadric is $\Psi = \underset{\sim}{\Gamma} \circ \varepsilon$.

Proof. Since $\underset{\sim}{\Sigma} \subseteq \underset{\sim}{\Gamma}$, the inclusions $\underset{\sim}{\Sigma}e \subseteq \underset{\sim}{\Gamma}e$ and $\underset{\sim}{\Sigma} \circ \varepsilon \subseteq \underset{\sim}{\Gamma} \circ \varepsilon$ are immediate. Since very element of $\underset{\sim}{\Gamma}$ is a product of invertible vectors, the opposite inclusions are a consequence of Lemma 4, its corollary, and induction. ■

Corollary. The group $\underset{\sim}{\Gamma}$ acts transitively on both S and Ψ. Moreover, these actions are equivalent with respect to the bijection from S to Ψ.

Proof. The actions are transitive, since the lemma establishes that S and Ψ are orbits of e and ε, respectively. By the bijection of Lemma 2, the elements $s = \underset{\sim}{\sigma}e$ and $\psi = \underset{\sim}{\sigma} \circ \varepsilon$ correspond. The element $\underset{\sim}{\gamma}$ of $\underset{\sim}{\Gamma}$ sends these to $\underset{\sim}{\gamma}s = \underset{\sim}{\gamma}(\underset{\sim}{\sigma}e) = (\underset{\sim}{\gamma}\underset{\sim}{\sigma})e$ and $\underset{\sim}{\gamma} \circ \psi = \underset{\sim}{\gamma} \circ (\underset{\sim}{\sigma} \circ \varepsilon) = (\underset{\sim}{\gamma}\underset{\sim}{\sigma}) \circ \varepsilon$, respectively. ■

Since corresponding elements of S and Ψ have the same stability subgroups in $\underset{\sim}{\Gamma}$, we may define the kernel $\underset{\sim}{Z}$ of the equivalent actions of $\underset{\sim}{\Gamma}$ on S and Ψ as the intersection of all the stability subgroups. This justifies defining the Möbius group G to be $\underset{\sim}{\Gamma}/\underset{\sim}{Z}$. The group G now acts effectively on both S and Ψ. Thus we have proved the following result.

Theorem. The Möbius group G acts transitively and effectively on both the projective spinors S and on the quadric Ψ and these actions are equivalent with respect to the bijection from S to Ψ.

Remark. The "units" of Γ are characterized as those γ in Γ such that $\gamma\mathbf{x} = \mathbf{x}\,{}^{I}\gamma$ for all $\mathbf{x}$ in X. Likewise, the "volume elements" of Γ are characterized by $\gamma\mathbf{x} = -\mathbf{x}\,{}^{I}\gamma$ for all $\mathbf{x}$ in X. Similar definitions hold for $\underset{\sim}{\Gamma}$. The subgroup $\underset{\sim}{Z}$ of $\underset{\sim}{\Gamma}$ consists of all such units and volume elements of $\underset{\sim}{\Gamma}$. These units are $\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)$ and the volume elements are

$$\begin{pmatrix} uk & 0 \\ 0 & -{}^{I}uk \end{pmatrix}$$

where u is some volume element of Γ and the k in $\mathring{K}$ are the units of Γ.

There are some interesting and nonstandard ways of characterizing the Clifford group of $\underset{\sim}{A}$. The following proposition shows the equivalence of these nonstandard characterizations to the standard ones.

Denote a typical element of $\underset{\sim}{A}$ by

$$\underset{\sim}{\alpha} = \begin{pmatrix} \alpha^{1} & \alpha^{\infty} \\ \alpha^{0} & \alpha^{2} \end{pmatrix}$$

and recall that $\underset{\sim}{\theta}$ is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Proposition 2. The following descriptions of the set of elements of the group $\underset{\sim}{\Gamma}$ are equivalent:

1. The product of invertible vectors of $\underset{\sim}{W}$.

2. Elements $\underset{\sim}{\alpha}$ in $\underset{\sim}{A}$ satisfying the "Vahlen conditions":

$$\underset{\sim}{\alpha}\,{}^J\underset{\sim}{\alpha}=\begin{pmatrix}k' & 0\\ 0 & k'\end{pmatrix}$$

where k' is in $\mathring{\underset{\sim}{K}}$, $\underset{\sim}{\alpha}\bar{e}$, ${}^J\underset{\sim}{\alpha}\bar{e}$, $\underset{\sim}{\alpha}\underset{\sim}{\theta}\bar{e}$, and ${}^J\underset{\sim}{\alpha}\underset{\sim}{\theta}\bar{e}$ are in $\bar{S}$.

3. In terms of the elements of Γ, elements of $\underset{\sim}{\Gamma}$ have one of the following simple forms:

(a) If the lower left entry of $\underset{\sim}{\alpha}$ is invertible, then

$$\underset{\sim}{\alpha}=\begin{pmatrix}\mathbf{a} & k+\mathbf{ab}\\ 1 & \mathbf{b}\end{pmatrix}\gamma$$

where $\mathbf{a}$ and $\mathbf{b}$ are in X, k is in $\mathring{\underset{\sim}{K}}$, and γ is in Γ.

(b) If some other entry of $\underset{\sim}{\alpha}$ is invertible, $\underset{\sim}{\alpha}$ has the above form changed only by permuting the appropriate rows or columns.

(c) If no entry of $\underset{\sim}{\alpha}$ is invertible, then

$$\underset{\sim}{\alpha}=\begin{pmatrix}\mathbf{z} & \mathbf{sz}\\ \mathbf{zt} & -\mathbf{szt}\end{pmatrix}\delta$$

where $\mathbf{z}$, $\mathbf{s}$, $\mathbf{t}$ are in X; $2\mathbf{s}\cdot\mathbf{z}$, $2\mathbf{t}\cdot\mathbf{z}$, $\mathbf{s}\cdot\mathbf{s}$, and $\mathbf{t}\cdot\mathbf{t}$ all equal 1; $\mathbf{z}\cdot\mathbf{z}=0$; and δ is in Γ.

Proof. The set described by 1 is contained in the set described by 2. This follows from Proposition 1.

The set described by 2 is contained in the set described by 3. In case that some entry of $\underset{\sim}{\alpha}$ is invertible, say the lower left entry; we have

$$\underset{\sim}{\alpha}=\begin{pmatrix}\mathbf{a} & k+\mathbf{ab}\\ 1 & \mathbf{b}\end{pmatrix}\gamma$$

where $\mathbf{a}$ and $\mathbf{b}$ are vectors, γ is the element α^0 of the Clifford group Γ, and k is the scalar $k=k'/\gamma^*\gamma$. That the first column is $\binom{\mathbf{a}}{1}\gamma$ and the second row is $(1\ \mathbf{b})\gamma$ follows from the (x)-case of Lemma 2 using the fact that $\underset{\sim}{\alpha}\bar{e}$ and ${}^J\underset{\sim}{\alpha}\bar{e}$, respectively, are in $\bar{S}$. That the second column is $\binom{k+\mathbf{ab}}{\mathbf{b}}\gamma$, follows from

$$\underset{\sim}{\alpha}\,{}^J\underset{\sim}{\alpha}=\begin{pmatrix}k' & 0\\ 0 & k'\end{pmatrix}$$

In the remaining case, when no entry of $\underset{\sim}{\alpha}$ is invertible, we have

$$\underset{\sim}{\alpha}=\begin{pmatrix}\mathbf{z} & \mathbf{sz}\\ \mathbf{zt} & -\mathbf{szt}\end{pmatrix}\delta$$

That the first column is $\binom{\mathbf{z}}{\mathbf{zt}}\delta$, that the first row is $(\mathbf{z}\ \mathbf{sz})\delta$, and that the second column is of the form $\binom{\mathbf{z}'}{\mathbf{z}'\mathbf{u}'}\delta'$ follow from the (z)-case of Lemma 2 using the fact that $\underset{\sim}{\alpha}\bar{e}$, ${}^J\underset{\sim}{\alpha}\underset{\sim}{\theta}\bar{e}$, and $\underset{\sim}{\alpha}\underset{\sim}{\theta}\bar{e}$, are, respectively, in $\bar{S}$. Here $\mathbf{z}'$ is an isotropic vector, $\mathbf{u}'$ is a vector such that $2\mathbf{u}'\cdot\mathbf{z}'=1$ and $\mathbf{u}'\cdot\mathbf{u}'=1$ and δ' is in Γ. This shows that

$$\underset{\sim}{\alpha} = \begin{pmatrix} \mathbf{z} & \mathbf{sz} \\ \mathbf{zt} & -\mathbf{szu} \end{pmatrix} \delta$$

where $\mathbf{u} = -\delta\delta'^{-1}u'\delta'\delta^{-1}$. The condition

$$\underset{\sim}{\alpha}^{J}\underset{\sim}{\alpha} = \begin{pmatrix} k' & 0 \\ 0 & k' \end{pmatrix}$$

implies the following conditions.

(i) (upper left entry)

$$\mathbf{zs}(2\mathbf{u}\cdot\mathbf{z}) + \mathbf{sz}(2\mathbf{t}\cdot\mathbf{z}) = k'$$

(ii) (lower left entry)

$$\mathbf{ztuzs} - \mathbf{szutz} = 2\mathbf{z} \wedge (\mathbf{t}(2\mathbf{u}\cdot\mathbf{z}) - \mathbf{u}(2\mathbf{t}\cdot\mathbf{z})) \wedge \mathbf{s} = 0$$

From (i), $2\mathbf{u}\cdot\mathbf{z} = 2\mathbf{t}\cdot\mathbf{z}$. This simplifies (ii) to $\mathbf{z} \wedge (\mathbf{t}-\mathbf{u}) \wedge \mathbf{s} = 0$. This then implies $\mathbf{t}-\mathbf{u} = \mathbf{z}l' + \mathbf{s}l$ for some l and l' in K. But $2\mathbf{s}\cdot\mathbf{z} \neq 0$, so $l=0$, and $\mathbf{u} = \mathbf{t} - \mathbf{z}l'$. Finally, $-\mathbf{szu} = -\mathbf{szt}$. Thus,

$$\underset{\sim}{\alpha} = \begin{pmatrix} \mathbf{z} & \mathbf{sz} \\ \mathbf{zt} & -\mathbf{szt} \end{pmatrix} \delta$$

The set described by 3 is contained in the set described by 1. It will be sufficient to consider the case

$$\underset{\sim}{\alpha} = \begin{pmatrix} \mathbf{a} & k + \mathbf{ab} \\ 1 & \mathbf{b} \end{pmatrix} \gamma$$

For, suppose it were instead of the form

$$\underset{\sim}{\alpha}' = \begin{pmatrix} \mathbf{z} & \mathbf{sz} \\ \mathbf{zt} & -\mathbf{szt} \end{pmatrix} \delta$$

Then we could choose $\mathbf{z}'$ to be any isotropic vector not orthogonal to $\mathbf{z}$ and set

$$\underset{\sim}{\beta} = \begin{pmatrix} \mathbf{z}+\mathbf{z}' & 1 \\ 0 & -(\mathbf{z}+\mathbf{z}') \end{pmatrix}$$

making $\underset{\sim}{\beta}$ an invertible vector in $\underset{\sim}{W}$. Then the upper left entry of $\underset{\sim}{\beta}\underset{\sim}{\alpha}'$ would be $\mathbf{z}'\mathbf{z} + \mathbf{zt}$, which would be in M by the corollary to Lemma $\tilde{0}$, and this would also be in Γ since

$$(\mathbf{z}'\mathbf{z} + \mathbf{zt})^*(\mathbf{z}'\mathbf{z} + \mathbf{zt}) = (2\mathbf{z}'\cdot\mathbf{z})(2\mathbf{t}\cdot\mathbf{z}) \neq 0$$

Since $\underset{\sim}{\theta}\underset{\sim}{\beta}\underset{\sim}{\alpha}'$ would have its lower left entry invertible, it would be of the desired form. Now it is easy to show

$$\begin{pmatrix} \mathbf{a} & k+\mathbf{ab} \\ 1 & \mathbf{b} \end{pmatrix} l^{-1} = \begin{pmatrix} \mathbf{a}+\mathbf{b}l & -\mathbf{a}\cdot\mathbf{a}+ml \\ 1-l & -(\mathbf{a}+\mathbf{b}l) \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{b} & m \\ -1 & -\mathbf{b} \end{pmatrix} \begin{pmatrix} \mathbf{a} & L \\ 1 & -\mathbf{a} \end{pmatrix}$$

where l is a nonzero scalar chosen so that $(\mathbf{a}+\mathbf{b})^2+k(l-1)/l$ is not zero, and the remaining scalars are defined by $m=-k/l+\mathbf{b}\cdot\mathbf{b}$ and

$$L=2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}+k(l-1)/l$$

Then $\mathbf{a}\cdot\mathbf{a}+L\neq 0$. The matrix

$$\begin{pmatrix} \mathbf{a}+\mathbf{b}l & -\mathbf{a}\cdot\mathbf{a}+ml \\ 1-l & -(\mathbf{a}+\mathbf{b}l) \end{pmatrix}$$

is invertible, since its square is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)(\mathbf{a}\cdot\mathbf{a}+L)l$. We see that

$$\underset{\sim}{\alpha} = \begin{pmatrix} \mathbf{a} & k+\mathbf{ab} \\ 1 & \mathbf{b} \end{pmatrix} l^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} l\gamma$$

is in $\underset{\sim}{\Gamma}$ by observing that for all γ in Γ, $\gamma=\mathbf{x}_1,\ldots,\mathbf{x}_r$, where the $\mathbf{x}_i$ are invertible vectors of X,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} l\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & l \\ l & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 & 0 \\ 0 & -\mathbf{x}_1 \end{pmatrix} \cdots \begin{pmatrix} \mathbf{x}_r & 0 \\ 0 & -\mathbf{x}_r \end{pmatrix} \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)^r$$

showing that $\underset{\sim}{\alpha}$ is also in $\underset{\sim}{\Gamma}$. ■

3. APPLICATIONS

3.1. Lie Geometry

Let

$$\underset{\sim}{w} = \begin{pmatrix} \mathbf{w} & -w^\infty \\ w^0 & -\mathbf{w} \end{pmatrix}$$

be in $\underset{\sim}{W}$, so that the associated equation of a "sphere" is

$$w^0\mathbf{x}\cdot\mathbf{x}-2\mathbf{w}\cdot\mathbf{x}+w^\infty=0$$

If $w^0\neq 0$ and $\underset{\sim}{w}\cdot\underset{\sim}{w}=-w^0w^\infty+\mathbf{w}\cdot\mathbf{w}\neq 0$, the sphere is a proper sphere and one calls $\underset{\sim}{w}\cdot\underset{\sim}{w}/(w^0)^2$ the square of its "radius."

If K were an ordered field in which positive elements had square roots, we would be able to write $\underset{\sim}{w}\cdot\underset{\sim}{w}=-w^0w^\infty+\mathbf{w}\cdot\mathbf{w}$ as

$$0=-w^0w^\infty-(w^r)^2+\mathbf{w}\cdot\mathbf{w}$$

where $\underset{\sim}{w}\cdot\underset{\sim}{w}$ is the square $(w^r)^2$. This motivates the following, where we make no special assumptions concerning square roots in K, but where we do require that K be an ordered field.

Let Y be an n-dimensional vector space equipped with the quadratic form $\mathbf{y} \rightsquigarrow \mathbf{y}\cdot\mathbf{y}$. Let X be the orthogonal sum of the two spaces $\mathbf{e}_r K$ and Y, where $\mathbf{e}_r\cdot\mathbf{e}_r = -1$. In this case, $\mathbf{x} = \mathbf{e}_r y^r + \mathbf{y}$ can be interpreted as representing an oriented sphere with center $\mathbf{y}$ and radius $|y^r|$. The sign of y^r determines the orientation.

To stress this new structure on X, we write a general element of $\underset{\sim}{W}$ as

$$\underset{\sim}{w} = \begin{pmatrix} \mathbf{e}_r w^r + \mathbf{w} & -w^{\infty} \\ w^0 & -(\mathbf{e}_r w^r + \mathbf{w}) \end{pmatrix}$$

Then, when $\underset{\sim}{w}\cdot\underset{\sim}{w} = -w^0 w^{\infty} - (w^r)^2 + \mathbf{w}\cdot\mathbf{w}$ is zero, $\underset{\sim}{w}$ is associated with the equation $w^0\mathbf{y}\cdot\mathbf{y} - 2\mathbf{w}\cdot\mathbf{y} + w^{\infty} = 0$ of a "sphere." Now the Möbius group G can be interpreted as the group of the Lie geometry of Y. The transformations represented by G do not act on the points of Y, but only act on the "spheres" of Y. Orthogonal "spheres" are not in general sent to orthogonal "spheres," but instead tangent oriented "spheres" are sent to tangent oriented "spheres." Thus, G is the spherical version of a Lie contact transformation. For more about Lie geometry over the reals see Yaglom (1981) and Rigby (1981).

3.2. Extended Action of Möbius Groups

Let us find the image of a "sphere" w of X under inversion in a proper "sphere" g of X. Let $\underset{\sim}{w}$ in $\underset{\sim}{W}$ represent w and let the invertible matrix $\underset{\sim}{\gamma}$ in $\underset{\sim}{W}$ represent g. Then the image is given by

$$\underset{\sim}{w}' = \frac{\underset{\sim}{\gamma} \circ \underset{\sim}{w}}{\underset{\sim}{\gamma}^* \underset{\sim}{\gamma}} = \frac{\underset{\sim}{\gamma}\underset{\sim}{w}^J \underset{\sim}{\gamma}}{\underset{\sim}{\gamma}^* \underset{\sim}{\gamma}} = \underset{\sim}{\gamma}\underset{\sim}{w}^I \underset{\sim}{\gamma}^{-1}$$

$$= (2\underset{\sim}{\gamma}\cdot\underset{\sim}{w} - \underset{\sim}{w}\underset{\sim}{\gamma})(-\underset{\sim}{\gamma}^{-1}) = \underset{\sim}{w} - 2\underset{\sim}{\gamma}\frac{\underset{\sim}{\gamma}\cdot\underset{\sim}{w}}{\underset{\sim}{\gamma}\cdot\underset{\sim}{\gamma}}$$

Here I and $*$ are the involution and anti-involution on $\underset{\sim}{A}$ that extend negation on $\underset{\sim}{W}$, respectively. Dividing by $\underset{\sim}{\gamma}^*\underset{\sim}{\gamma}$ was not necessary to obtain the equation of the image "sphere," but makes the transformation orthogonal and more easily recognized simply as a "reflection."

Introduce a new linear space $\bar{W}$ which is isometric to the space $\underset{\sim}{W}$ and represents equations of "spheres" of X. The isometry is given by

$$\bar{w} = \begin{pmatrix} w^0 \\ w^{\infty} \\ \mathbf{w} \end{pmatrix} \rightsquigarrow \underset{\sim}{w} = \begin{pmatrix} \mathbf{w} & -w^{\infty} \\ w^0 & -\mathbf{w} \end{pmatrix}$$

when $\bar{W}$ is equipped with the quadratic form $\bar{w} \rightsquigarrow \bar{w}\cdot\bar{w} = -w^0 w^{\infty} + \mathbf{w}\cdot\mathbf{w}$. So $\bar{w}$ and $\underset{\sim}{w}$ both represent the equation $w^0\mathbf{x}\cdot\mathbf{x} - 2\mathbf{w}\cdot\mathbf{x} + w^{\infty}$. Transporting the transformation from $\underset{\sim}{W}$ to $\bar{W}$, we obtain $\bar{w}' = \bar{w} - 2\bar{\gamma}(\bar{\gamma}\cdot\bar{w}/\bar{\gamma}\cdot\bar{\gamma})$. Now $\bar{w}'$ represents the equation of the image "sphere."

This interpretation is by far the simplest way to calculate the equation of the image of a "sphere" represented by $\underset{\sim}{w}$ with respect to inversion in the proper "sphere" represented by $\underset{\sim}{\gamma}$. Compare this with Yaglom (1981), p. 352.

Finally, since the transformations $\bar{w}' = \bar{w} - 2\bar{\gamma}(\bar{\gamma}\cdot\bar{w}/\bar{\gamma}\cdot\bar{\gamma})$, for all invertible $\underset{\sim}{\gamma}$ in $\underset{\sim}{W}$, generate the orthogonal group $O(\bar{W})$, the methods of this paper yield in a unified way (1) the Möbius group of X, (2) the group of the Lie geometry of Y; and (3) the orthogonal group of $\bar{W}$, as well as the natural extensions that act on (1) the "spheres" of X and (2) what is classically known as bundles of "spheres" that touch a given "sphere."

REFERENCES

Ahlfors, L. V. (1986). Möbius transformations in R^n expressed through 2×2 matrices of Clifford numbers, *Complex Variables, Theory and Applications*, **5**, 215-224.

Cartan, E. (1937). *Leçons sur la théorie des spineurs*, Paris.

Cartan, E. (1981). *The Theory of Spinors*, Dover, New York.

Conway, J. H., *et al.* (1985). *Atlas of Finite Groups*, Clarendon Press, Oxford.

Dirac, P. A. M. (1927). The quantum theory of the electron, *Proceedings of the Royal Society (A)*, **117**, 610-624.

Dirac, P. A. M. (1928). The quantum theory of the electron, *Proceedings of the Royal Society (A)*, **118**, 351-362.

Hua, L. K. (1981). *Starting with the Unit Circle: Background to Higher Analysis*, Springer-Verlag, New York.

Klein, F. (1926). *Vorlesungen über höhere Geometrie*, 3rd ed. Springer-Verlag, Berlin [reprinted, Chelsea, New York (1957)].

Lam, T. Y. (1973). *The Algebraic Theory of Quadratic Forms*, W. A. Benjamin, Reading, Massachusetts.

Lounesto, P., and Latvamaa, E. (1980). Conformal transformations and Clifford algebras, *Proceedings of the American Mathematical Society*, **79**, 533-538.

Lounesto, P., and Springer, A. (1989). Möbius transformations and Clifford algebras of Euclidean and anti-Euclidean spaces, in *Deformations of Mathematical Structures*, J. Ławrynowicz, ed., Kluwer Academic Publishers, Dordrecht, pp. 79-90.

Penrose, R., and Rindler, W. (1984). *Spinors and Space-Time*, Cambridge University Press, Cambridge.

Rigby, J. F. (1981). The geometry of cycles, and generalized Laguerre inversions, in *The Geometric Vein*, C. Davis, ed., Springer-Verlag, New York, pp. 355-378.

Yaglom, I. M. (1981). Circular transformations of Möbius, Laguerre, and Lie, in *The Geometric Vein*, C. Davis, ed., Springer-Verlag, New York, pp. 345-353.